8.3 - Nonhomogeneous Linear Systems

$\mathbf{X}' = \mathbf{AX} + \mathbf{F}(t)$ is a nonhomogeneous system if $\mathbf{F}(t) \neq \mathbf{0}$.

Example: Use the method of undetermined coefficients to solve the given nonhomogeneous system.

$$\mathbf{X}' = \begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 4t + 9e^{6t} \\ -t + e^{6t} \end{pmatrix}$$

Find $\vec{X}_c$, the solution to the associated homogeneous system.

$$\begin{vmatrix} 1-\lambda & -4 \\ 4 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda-1)^2 + 16 = 0$$

$$(\lambda-1)^2 = -16 \Rightarrow \lambda = 1 \pm 4i$$

$$\begin{pmatrix} -4i & -4 \\ 4 & -4i \end{pmatrix} \vec{K} = \vec{0} \quad \text{gives} \quad 4k_1 = 4ik_2$$

If $k_2 = -i$, then $k_1 = 1 \Rightarrow \vec{K} = \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} i$

$$\vec{X}_c = c_1 e^t \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos 4t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin 4t \right] + c_2 e^t \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos 4t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin 4t \right]$$

$$\vec{X}_c = \begin{pmatrix} e^t \cos 4t \\ e^t \sin 4t \end{pmatrix} c_1 + \begin{pmatrix} e^t \sin 4t \\ -e^t \cos 4t \end{pmatrix} c_2 = \underbrace{\begin{pmatrix} e^t \cos 4t & e^t \sin 4t \\ e^t \sin 4t & -e^t \cos 4t \end{pmatrix}}_{\text{Fundamental matrix}} \vec{C}$$

$$\vec{X}_c = \Phi \vec{C}$$

phi

$$\vec{F}(t) = \begin{pmatrix} 4t + 9e^{6t} \\ -t + e^{6t} \end{pmatrix} = \underbrace{\begin{pmatrix} 4 \\ -1 \end{pmatrix} t}_{\text{linear}} + \underbrace{\begin{pmatrix} 9 \\ 1 \end{pmatrix} e^{6t}}_{\text{exponential}}$$

form: $At + B$

$$\vec{X}_p = \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} t + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} e^{6t} \quad \text{into} \quad \vec{X}' = A\vec{X} + \vec{F}$$

$$\begin{pmatrix} a_3 \\ b_3 \end{pmatrix} + \begin{pmatrix} 6a_1 \\ 6b_1 \end{pmatrix} e^{6t} = \begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} a_3 t + a_2 + a_1 e^{6t} \\ b_3 t + b_2 + b_1 e^{6t} \end{pmatrix} + \begin{pmatrix} 4t + 9e^{6t} \\ -t + e^{6t} \end{pmatrix}$$

$$\begin{pmatrix} a_3 + 6a_1e^{6t} \\ b_3 + 6b_1e^{6t} \end{pmatrix} = \begin{pmatrix} a_3t + a_2 + a_1e^{6t} - 4b_3t - 4b_2 - 4b_1e^{6t} + 4t + 9e^{6t} \\ 4a_3t + 4a_2 + 4a_1e^{6t} + b_3t + b_2 + b_1e^{6t} - t + e^{6t} \end{pmatrix}$$

e^{6t}: $6a_1 = a_1 - 4b_1 + 9$ Const: $a_3 = a_2 - 4b_2$

$6b_1 = 4a_1 + b_1 + 1$ $b_3 = 4a_2 + b_2$

t terms: $0 = a_3 - 4b_3 + 4$

$0 = 4a_3 + b_3 - 1$

AALOW:

$$\boxed{\vec{x} = c_1e^t\begin{pmatrix} \cos 4t \\ \sin 4t \end{pmatrix} + c_2e^t\begin{pmatrix} \sin 4t \\ -\cos 4t \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}t + \begin{pmatrix} 4/17 \\ 1/17 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{6t}}$$

Undetermined coeff's is of particularly limited use - functions can only be const poly, sine/cosine exp, +, × and redundancy doesn't work how we've seen

Variation of parameters

The complementary solution is $\mathbf{X}_c(t) = \Phi(t)\mathbf{C}$.

Definition: $\Phi(t)$ is the **fundamental matrix** of the system.

For a system with independent solutions, the Wronskian is nonzero, so Φ^{-1} exists. Since Φ is a solution to $\mathbf{X}' = \mathbf{A}\mathbf{X}$, we have $\Phi' = \mathbf{A}\Phi$. Suppose $\mathbf{X}_p = \Phi(t)\mathbf{U}(t)$, where $\mathbf{U}(t)$ is a column matrix of unknown functions.

$2x + 3y = 5$

$4x - 8y = 7$

Order matters for matrix multiplication $A^{-1}A\vec{x} = A^{-1}\vec{b}$

$\vec{X}_p' = \Phi'\vec{u} + \Phi\vec{u}'$ into $\vec{x}' = A\vec{x} + \vec{F}$

gives $\cancel{\Phi'\vec{u}} + \Phi\vec{u}' = \cancel{A\Phi\vec{u}} + \vec{F}$ but

$\Phi' = A\Phi$ $\Rightarrow \Phi\vec{u}' = \vec{F} \Rightarrow \vec{u}' = \Phi^{-1}\vec{F}$

$$\vec{u} = \int \Phi^{-1} \vec{F}(t)\, dt \quad \text{and}$$

$$\vec{X}_p = \Phi \vec{u}(t)$$

$$\overbrace{\mathbf{X}(t) = \Phi(t)\mathbf{C}}^{\vec{X}_c} + \overbrace{\underbrace{\Phi(t)}_{\Phi} \int \underbrace{\Phi^{-1}(t)\mathbf{F}(t)\, dt}_{\vec{u}}}^{\vec{X}_p}$$

Example: Use variation of parameters to solve the given nonhomogeneous system.

$$\frac{dx}{dt} = 2x - y$$

$$\frac{dy}{dt} = 3x - 2y + 4t$$

$$\Rightarrow \vec{X}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \vec{X} + \begin{pmatrix} 0 \\ 4t \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 4 + 3 = 0 \Rightarrow \lambda = \pm 1$$

$$\lambda_1 = 1 \quad \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \Rightarrow \vec{K}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1 \quad \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \Rightarrow \vec{K}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$3k_1 - k_2 = 0$

$3k_1 = k_2$

$$\vec{X}_c = \underbrace{\begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}}_{\Phi} \vec{C} \text{ because } \vec{X}_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$$

$$= \begin{pmatrix} e^t \\ e^t \end{pmatrix} c_1 + \begin{pmatrix} e^{-t} \\ 3e^{-t} \end{pmatrix} c_2$$

$$= \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Matrix inverses:

2x2

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \left(AA^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$\Phi^{-1} = \frac{1}{2} \begin{pmatrix} 3e^{-t} & -e^{-t} \\ -e^t & e^t \end{pmatrix}$$

$$\Phi^{-1}\vec{F} = \frac{1}{2}\begin{pmatrix} 3e^{-t} & -e^{-t} \\ -e^{t} & e^{t} \end{pmatrix}\begin{pmatrix} 0 \\ 4t \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -4te^{-t} \\ 4te^{t} \end{pmatrix} = \begin{pmatrix} -2t\,e^{-t} \\ 2t\,e^{t} \end{pmatrix}$$

$$\int \Phi^{-1}\vec{F}\,dt = \int \begin{pmatrix} -2te^{-t} \\ 2te^{t} \end{pmatrix} dt = \begin{pmatrix} 2te^{-t} + 2e^{-t} \\ 2te^{t} - 2e^{t} \end{pmatrix}$$

$$\Phi \int \Phi^{-1}\vec{F}\,dt = \begin{pmatrix} e^{t} & e^{-t} \\ e^{t} & 3e^{-t} \end{pmatrix}\begin{pmatrix} 2te^{-t} + 2e^{-t} \\ 2te^{t} - 2e^{t} \end{pmatrix}$$

$$= \begin{pmatrix} 2t + \cancel{2} + 2t - \cancel{2} \\ 2t + 2 + 6t - 6 \end{pmatrix} = \begin{pmatrix} 4t \\ 8t - 4 \end{pmatrix}$$

$$\boxed{\vec{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-t} + \begin{pmatrix} 4 \\ 8 \end{pmatrix}t + \begin{pmatrix} 0 \\ -4 \end{pmatrix}}$$

Example: Use variation of parameters to solve the given nonhomogeneous system.

$$\mathbf{X}' = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix} \mathbf{X} + \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix} e^{2t}$$

$\lambda \to \vec{K} \to \vec{X}_c \to \Phi$

$\vec{X}_p = \Phi \int \Phi^{-1} \vec{F}\, dt$

$\begin{vmatrix} 2-\lambda & -1 \\ 4 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 4\lambda + 4 + 4 = 0 \Rightarrow \lambda^2 - 4\lambda = -8$

$\lambda^2 - 4\lambda + 4 = -8 + 4 \Rightarrow \lambda = 2 \pm 2i$

$\lambda = 2+2i: \begin{pmatrix} -2i & -1 \\ 4 & -2i \end{pmatrix} \to -2ik_1 = k_2$ Let $k_1 = 1$.

$\vec{K} = \begin{pmatrix} 1 \\ -2i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} i$

$\vec{X}_c = c_1 e^{2t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos 2t - \begin{pmatrix} 0 \\ -2 \end{pmatrix}\sin 2t\right] + c_2 e^{2t}\left[\begin{pmatrix} 0 \\ -2 \end{pmatrix}\cos 2t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin 2t\right]$

$\vec{X}_c = \underbrace{\boxed{\begin{pmatrix} e^{2t}\cos 2t & e^{2t}\sin 2t \\ 2e^{2t}\sin 2t & -2e^{2t}\cos 2t \end{pmatrix}}}_{\Phi} \vec{C}$

$\to \Phi = e^{2t}\begin{pmatrix} \cos 2t & \sin 2t \\ 2\sin 2t & -2\cos 2t \end{pmatrix}$

$\Phi^{-1} = -\frac{1}{2e^{4t}}\begin{pmatrix} -2e^{2t}\cos 2t & -e^{2t}\sin 2t \\ -2e^{2t}\sin 2t & e^{2t}\cos 2t \end{pmatrix}$

$\Phi^{-1}\vec{F} = \begin{pmatrix} e^{-2t}\cos 2t & \frac{1}{2}e^{-2t}\sin 2t \\ e^{-2t}\sin 2t & -\frac{1}{2}e^{-2t}\cos 2t \end{pmatrix}\begin{pmatrix} e^{2t}\sin 2t \\ 2e^{2t}\cos 2t \end{pmatrix}$

$= \begin{pmatrix} \cos 2t \sin 2t + \cos 2t \sin 2t \\ \sin^2 2t - \cos^2 2t \end{pmatrix} = \begin{pmatrix} \sin 4t \\ -\cos 4t \end{pmatrix}$

$$\int \Phi^{-1}\vec{F}\,dt = \int \begin{pmatrix} \sin 4t \\ -\cos 4t \end{pmatrix} dt = \begin{pmatrix} -\frac{1}{4}\cos 4t \\ -\frac{1}{4}\sin 4t \end{pmatrix}$$

$$\Phi \int \Phi^{-1}\vec{F}\,dt = -\frac{e^{2t}}{4}\begin{pmatrix} \cos 2t & \sin 2t \\ 2\sin 2t & -2\cos 2t \end{pmatrix}\begin{pmatrix} \cos 4t \\ \sin 4t \end{pmatrix}$$

$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta)$

$$= -\frac{e^{2t}}{4}\begin{pmatrix} \cos 2t\cos 4t + \sin 2t \sin 4t \\ 2\sin 2t\cos 4t - 2\cos 2t \sin 4t \end{pmatrix}$$

$$= -\frac{e^{2t}}{4}\begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}\cos 2t \\ \frac{1}{2}\sin 2t \end{pmatrix} e^{2t}$$

$$\boxed{\vec{X} = c_1\begin{pmatrix} \cos 2t \\ 2\sin 2t \end{pmatrix} e^{2t} + c_2\begin{pmatrix} \sin 2t \\ -2\cos 2t \end{pmatrix} e^{t} + \begin{pmatrix} -\frac{1}{4}\cos 2t \\ \frac{1}{2}\sin 2t \end{pmatrix} e^{2t}}$$

Alternative for finding A^{-1}:

$$(A \mid I) \xrightarrow[\text{ops}]{\text{row}} \left(\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \middle| \; A^{-1}\right)$$

$x(t_0) = 3,\ y(t_0) = 5 \quad \rightarrow \quad \begin{pmatrix} x(t_0) \\ y(t_0) \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

For an initial-value problem of the form $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t)$, $X(t_0) = \mathbf{X}_0$, we have $\mathbf{X} = \Phi\mathbf{C} + \Phi\int_{t_0}^{t}\Phi^{-1}(s)\mathbf{F}(s)\,ds \implies \mathbf{X}(t_0) = \Phi(t_0)\mathbf{C} \implies \mathbf{C} = \Phi^{-1}(t_0)\mathbf{X}(t_0)$.
Thus,

$$\mathbf{X}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{X}(t_0) + \Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)\mathbf{F}(s)\,ds$$

Example: Solve the given initial-value problem.

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 1/t \\ 1/t \end{pmatrix}, \quad \mathbf{X}(1) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$t_0 = 1$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 + 1 = 0 \Rightarrow \lambda = 0, \text{ repeated}$$

$$\lambda = 0 : \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \to \vec{K} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(A - \lambda I)\vec{P} = \vec{K} \Rightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$p_1 - p_2 = 1. \quad \text{Let } p_2 = 0 \Rightarrow \vec{P} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{X}_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$$

$$\vec{X}_c = \begin{pmatrix} 1 & t+1 \\ 1 & t \end{pmatrix} \vec{C} \Rightarrow \Phi = \begin{pmatrix} 1 & t+1 \\ 1 & t \end{pmatrix}$$

$$\Phi^{-1} = \frac{1}{-1} \begin{pmatrix} t & -(t+1) \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -t & t+1 \\ 1 & -1 \end{pmatrix}$$

$$\Phi^{-1} \vec{F} = \begin{pmatrix} -t & t+1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1/t \\ 1/t \end{pmatrix} = \begin{pmatrix} 1/t \\ 0 \end{pmatrix}$$

$$\int_{t_0}^{t} \Phi^{-1} \vec{F}(s)\, ds = \int_1^t \begin{pmatrix} 1/s \\ 0 \end{pmatrix} ds = \begin{pmatrix} \ln s \\ 0 \end{pmatrix} \Bigg|_1^t$$

$$= \begin{pmatrix} \ln t - \ln 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \ln t \\ 0 \end{pmatrix}$$

$\vec{X}_p$

$$\Phi \int_{t_0}^{t} \Phi^{-1} F(s)\, ds = \begin{pmatrix} 1 & t+1 \\ 1 & t \end{pmatrix} \begin{pmatrix} \ln t \\ 0 \end{pmatrix} = \begin{pmatrix} \ln t \\ \ln t \end{pmatrix}$$

$$\Phi^{-1}(t_0)\, \underbrace{\vec{X}(t_0)}_{\text{given}} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$t_0 = 1$

$$\Phi^{-1}(t) = \begin{pmatrix} -t & t+1 \\ 1 & -1 \end{pmatrix}$$

$$\vec{X}(t) = \begin{pmatrix} 1 & t+1 \\ 1 & t \end{pmatrix}\begin{pmatrix} -4 \\ 3 \end{pmatrix} + \begin{pmatrix} \ln t \\ \ln t \end{pmatrix}$$

$$\boxed{\vec{X}(t) = \begin{pmatrix} 3t-1 \\ 3t-4 \end{pmatrix} + \begin{pmatrix} \ln t \\ \ln t \end{pmatrix}}$$

$$\left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\, I \mid A^{-1} \right.$$

$$A = \begin{array}{ccc} 11 & 12 & 13 \\ + & - & + \\ \end{array}$$
$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

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Minor cofactor : C_{ij}

$$a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = \det(A)$$

$$[C_{ij}]^T$$

$$A^T = \begin{bmatrix} | & | & | \\ | & | & | \\ | & | & | \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)}[C_{ij}]^T$$